MAXIMUM PRINCIPLE AND SYMMETRY FOR MINIMAL HYPERSURFACES IN $\mathbb{H}^n \times \mathbb{R}$

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Abstract. The aim of this work is to study how the asymptotic boundary of a minimal hypersurface in $\mathbb{H}^n \times \mathbb{R}$ determines the behavior of the hypersurface at finite points, in several geometric situations.

1. Introduction

In this article we discuss how, in several geometric situations, the shape at infinity of a minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ determines the shape of the surface itself.
A beautiful theorem in minimal surfaces theory is the Schoen's characterization of the catenoid [12]. It can be stated as follows. *Let $M \subset \mathbb{R}^3$ be a complete immersed minimal surface with two annular ends. Assume that each end is a graph, then M is a catenoid.* On the other hand, there exists a complete minimal annulus immersed in a slab of $\mathbb{R}^3$ [6].
A characterization of the catenoid in the hyperbolic space, assuming regularity at infinity, was established by G. Levitt and H. Rosenberg in [5]. In a joint work with L. Hauswirth [3], the authors of the present article proved a Schoen type theorem in $\mathbb{H}^2 \times \mathbb{R}$, in the class of finite total curvature surfaces.

Our first result is a new Schoen type theorem in $\mathbb{H}^2 \times \mathbb{R}$. Namely, we replace Schoen's assumption *each end is a graph* with the assumption *each end is a vertical graph whose asymptotic boundary is a copy of the asymptotic boundary of* $\mathbb{H}^2$ (Theorem 2.1).

Our second result is a *maximum principle* in a vertical (closed) halfspace. Assume that M is a complete minimal surface, possibly with finite boundary, properly immersed in $\mathbb{H}^2 \times \mathbb{R}$ and that the boundary of M, if any, is contained in the closure of a vertical halfspace P_+. Assume further that the points at finite height of the asymptotic boundary of M are contained in the asymptotic boundary of the halfspace P_+. Then M is entirely contained in the halfspace P_+, unless M is equal to the vertical halfplane ∂P_+ (Theorem 3.1).

Then we generalize our results to higher dimensions.

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Theorem 2.1 and Theorem 3.1 in higher dimension are analogous to the 2-dimensional case. In order to generalize Theorem 2.1, we first need to give a characterization of the n-catenoid analogous to that of the 2-dimensional case (Theorem 4.2, see also [2]).
Moreover in the higher dimensional case, it is worthwhile to state some interesting consequences of our results.
Let S_∞ be a closed set contained in an open slab of $\partial_\infty \mathbb{H}^n \times \mathbb{R}$ with height equal to $\pi/(n-1)$ such that the projection of S_∞ on $\partial_\infty \mathbb{H}^n \times \{0\}$ omits an open subset.
We prove that there is no complete properly immersed minimal hypersurface M whose asymptotic boundary is S_∞ (Theorem 4.5-(2)).
Finally we prove an Asymptotic Theorem (Theorem 4.6), that implies the following non-existence result. There is no horizontal minimal graph over a bounded strictly convex domain, see [9, Equation (3)], given by a positive function g continuous up to the boundary, taking zero boundary value data (Remark 4.1).

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2. A characterization of the catenoid in $\mathbb{H}^2 \times \mathbb{R}$

We are going to prove the characterization of the catenoid presented in the Introduction. Any surface in $\mathbb{H}^2 \times \mathbb{R}$ with constant third coordinate is a complete totally geodesic surface called a *slice*. For any $s \in \mathbb{R}$, we denote by Π_s the slice $\mathbb{H}^2 \times \{s\}$ and we set $\Pi_s^+ = \{(p,t) \mid p \in \mathbb{H}^2,\ t > s\}$ and $\Pi_s^- = \{(p,t) \mid p \in \mathbb{H}^2,\ t < s\}$. For simplicity Π stands for Π_0.

Lemma 2.1. *Let Γ^+ and Γ^- be two Jordan curves in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$ which are vertical graphs over $\partial_\infty \mathbb{H}^2 \times \{0\}$ and such that $\Gamma^+ \subset \partial_\infty \Pi^+$ and $\Gamma^- \subset \partial_\infty \Pi^-$. Assume that Γ^- is the symmetry of Γ^+ with respect to Π.*
Let $M \subset \mathbb{H}^2 \times \mathbb{R}$ be an immersed, connected, complete minimal surface with two ends E^+ and E^-. Assume that each end is a vertical graph and that $\partial_\infty M = \Gamma^+ \cup \Gamma^-$, that is $\partial_\infty E^+ = \Gamma^+$ and $\partial_\infty E^- = \Gamma^-$.
Then M is symmetric with respect to Π. Furthermore, each part $M \cap \Pi^\pm$ is a vertical graph and M is embedded.

Proof. For any $t > 0$ we set $M_t^+ = M \cap \Pi_t^+$. We denote by M_t^{+*} the symmetry of M_t^+ with respect to the slice Π_t. Furthermore, we denote by t^+ the highest t-coordinate of Γ^+. Since $\partial_\infty M = \Gamma^+ \cup \Gamma^-$, then $M \cap \Pi_{t^+} = \emptyset$, by the maximum principle.
We denote by E^+ the end of M whose asymptotic boundary is Γ^+. As E^+ is a vertical graph, there exists $\varepsilon > 0$ such that $M_{t^+-\varepsilon}^+$ is a vertical graph, then we can start Alexandrov reflection [1].

We keep doing the Alexandrov reflection with Π_t, doing $t \searrow 0$. By applying the interior or boundary maximum principle, we get that, for $t > 0$, the surface M_t^{+*} stays above M_t^-. Therefore we get that M_0^+ is a vertical graph and that M_0^{+*} stays above M_0^-. Doing Alexandrov reflection with slices coming from below, one has that M_0^- is a vertical graph and that M_0^{-*} stays below M_0^+, henceforth we get $M_0^{+*} = M_0^-$. Thus M is symmetric with respect to Π and each component of $M \setminus \Pi$ is a graph. Therefore we can show, as in the proof of [12, Theorem 2], that the whole surface M is embedded. This completes the proof. $\square$

Definition 2.1. A *vertical plane* is a complete totally geodesic surface $\gamma \times \mathbb{R}$ where γ is any complete geodesic of $\mathbb{H}^2$.

Theorem 2.1. *Let $M \subset \mathbb{H}^2 \times \mathbb{R}$ be an immersed, connected, complete minimal surface with two ends. Assume that each end is a vertical graph whose asymptotic boundary is a copy of $\partial_\infty \mathbb{H}^2$. Then M is rotational, hence M is a catenoid.*

Proof. Up to a vertical translation, we can assume that the asymptotic boundary is symmetric with respect to the slice Π. We use the same notations as in the proof of Lemma 2.1. We know from Lemma 2.1 that M is symmetric with respect to Π and that M_0^+ and M_0^- are vertical graphs. Therefore, at any point of $M \cap \Pi$ the tangent plane of M is orthogonal to Π.
We have $\partial_\infty M = \partial_\infty \mathbb{H}^2 \times \{t_0, -t_0\}$ for some $t_0 > 0$. Since M is embedded, M separates $\mathbb{H}^2 \times [-t_0, t_0]$ into two connected components. We denote by U_1 the component whose asymptotic boundary is $\partial_\infty \mathbb{H}^2 \times [-t_0, t_0]$ and by U_2 the component such that $\partial_\infty U_2 = \partial_\infty \mathbb{H}^2 \times \{t_0, -t_0\}$.
Let $q_\infty \in \partial_\infty \mathbb{H}^2$ and let $\gamma \subset \mathbb{H}^2$ be an oriented geodesic issuing from q_∞, that is $q_\infty \in \partial_\infty \gamma$. Let $q_0 \in \gamma$ be any fixed point.
For any $s \in \mathbb{R}$, we denote by P_s the vertical plane orthogonal to γ passing through the point of γ whose oriented distance from q_0 is s. We suppose that $s < 0$ for any point in the geodesic segment (q_0, q_∞).
For any $s \in \mathbb{R}$, we call $M_s(l)$ the part of $M \setminus P_s$ such that $(q_\infty, t_0), (q_\infty, -t_0) \in \partial_\infty M_s(l)$ and let $M_s^*(l)$ be the reflection of $M_s(l)$ about P_s. We denote by $M_s(r)$ the other part of $M \setminus P_s$ and by $M_s^*(r)$ its reflection about P_s.

It will be clear from the following two Claims, why we can start the Alexandrov reflection principle with respect to the vertical planes P_s and obtain the result.

By assumptions there exists $s_1 < 0$ such that for any $s < s_1$ the part $M_s(l)$ has two connected components and both of them are vertical graphs. We deduce that $\partial M_s(l)$ has two (symmetric) connected components, each one being a vertical graph.

We recall that $\Pi^+ := \{t > 0\}$ and $\Pi^- := \{t < 0\}$.

Claim 1. For any $s < s_1$, we have that $M_s^*(l) \cap \Pi^+$ stays above $M_s(r)$ and $M_s^*(l) \cap \Pi^-$ stays below $M_s(r)$. Consequently $M_s^*(l) \subset U_2$ for any $s < s_1$.

Observe that $M_s^*(l) \cap \Pi^+$ and $M_s(r) \cap \Pi^+$ have same asymptotic boundary and that $\partial\left(M_s^*(l) \cap \Pi^+\right) = \partial M_s(r) \cap \Pi^+$. Therefore the asymptotic and finite boundaries of any

lifting up of $M_s^*(l)$ is above the asymptotic and finite boundaries of $M_s(r)$. Hence any lifting up of $M_s^*(l)$ is above $M_s(r)$ by the maximum principle, which ensures that the whole $M_s^*(l) \cap \Pi^+$ stays above $M_s(r)$ for any $s < s_1$, as desired. The proof of the other assertion is analogous. Then, Claim 1 is proved.

We set

$$\sigma = \sup \big\{ s \in \mathbb{R} \mid M_t^*(l) \cap \Pi^+ \text{ stays above } M_t(r) \cap \Pi^+ \text{ for any } t \in (-\infty, s) \big\}.$$

Claim 2. We have $M_\sigma^*(l) = M_\sigma(r)$. Thus, given a geodesic $\gamma \subset \mathbb{H}^2$, there exists a vertical plane P_σ orthogonal to γ such that M is symmetric with respect to P_σ

Note that we also have

$$\sigma = \sup \big\{ s \in \mathbb{R} \mid M_t^*(l) \subset U_2 \text{ for any } t \in (-\infty, s) \big\}.$$

In order to prove Claim 2, we first establish the following fact.

Assertion. For any s such that $M_s^*(l) \cap \Pi \subset U_2$ then $M_s^*(l) \subset U_2$.

As M is symmetric with respect to Π the intersection $M \cap \Pi$ is constituted of a finite number of pairwise disjoint Jordan curves $C_1, \dots, C_k$. Since $M \cap \Pi^+$ is a vertical graph we deduce

$$(C_j \times \mathbb{R}) \cap M = C_j \quad \text{for any } j = 1, \dots, k.$$

Moreover, since M is connected and is symmetric about Π, we get that $M \cap \Pi^+$ is connected.

Let $D_j \subset \Pi$ be the Jordan domain bounded by C_j, $j = 1, \dots, k$. Noticing that:

- $(M \cap \Pi^+) \setminus (\overline{D}_j \times \mathbb{R}) \neq \emptyset$,
- $M \cap \Pi^+$ is connected,
- $M \cap (C_j \times \mathbb{R}) = C_j$,
- $\partial_\infty M \cap \Pi^+ = \partial_\infty \mathbb{H}^2 \times \{t_0\}$,

we get that $(M \cap \Pi^+) \cap (D_j \times \mathbb{R}) = \emptyset$, $j = 1, \dots, k$. Hence, $D_i \cap D_j = \emptyset$ for any $i \neq j$. Therefore, $M \cap \Pi^+$ is a vertical graph over $\Pi \setminus \cup D_i$.

This implies that, for any $\varepsilon > 0$, the vertical translation $(M_s^*(l) \cap \Pi^+) + (0, 0, \varepsilon)$ stays above M. This proves the Assertion.

Let us continue the proof of Claim 2. The definition of σ implies that $M_{\sigma+\varepsilon}^*(l) \cap U_1 \neq \emptyset$, for ε small enough.

We deduce from the Assertion that $M_{\sigma+\varepsilon}^*(l) \cap \Pi$ is not contained in U_2 for any small enough $\varepsilon > 0$. Hence we infer that $M_\sigma^*(l) \cap \Pi$ and $M_\sigma(r) \cap \Pi$ are tangent at an interior or boundary point lying in some Jordan curve C_j contained in $M \cap \Pi$. Since $M_\sigma^*(l) \subset \overline{U}_2$, $M_\sigma(r) \subset \partial U_2$ and the tangent plane of M is vertical along $M \cap \Pi$, we are able to apply the maximum principle (possibly with boundary) to conclude that $M_\sigma^*(l) = M_\sigma(r)$, that is P_σ is a plane of symmetry of M. This proves Claim 2.

For any $\alpha \in (0, \pi/2]$ consider a family of vertical planes making an angle α with P_σ, generated by hyperbolic translations along the horizontal geodesic $P_\sigma \cap \Pi$. Now, doing the Alexandrov reflection principle with this family of planes, we find a vertical plane of symmetry of M, say P^α. Hence M is invariant by the rotation of angle 2α around

the vertical geodesic $P^\alpha \cap P_\sigma$. Choosing an angle α such that π/α is not rational, we find that M is invariant by rotation around the axis $P^\alpha \cap P_\sigma$. This concludes the proof of Theorem 2.1, as desired. $\square$

Remark 2.1. For any integer n, there exists a minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ which is a vertical graph, whose asymptotic boundary is a copy of $\partial_\infty \mathbb{H}^2$ and whose finite boundary is constituted of n smooth Jordan curves in the slice Π, see [10, Theorem 5.1]. In the same article the second and third author asked about the existence of such graphs with two boundary curves in Π cutting orthogonally the slice Π. Theorem 2.1 implies that the answer to this question is negative.

3. Maximum Principle in a vertical halfspace of $\mathbb{H}^2 \times \mathbb{R}$.

In this section we prove some maximum principle in a vertical halfspace. More precisely, we prove that, under some geometric assumptions, the behavior of the asymptotic boundary of M at finite height, determines the behaviour of M.

Definition 3.1. We call a *vertical halfspace* any of the two components of $(\mathbb{H}^2 \times \mathbb{R}) \setminus P$, where P is a vertical plane.

Theorem 3.1. *Let M be a complete minimal surface, possibly with finite boundary, properly immersed in $\mathbb{H}^2 \times \mathbb{R}$. Let P be a vertical plane and let P_+ be one of the two halfspaces determined by P. If $\partial M \subset \overline{P_+}$ and $\partial_\infty M \cap (\partial_\infty \mathbb{H}^2 \times \mathbb{R}) \subset \partial_\infty P_+$, then $M \setminus \partial M \subset P_+$, unless $M \subset P$.*

For the proof of Theorem 3.1 we need to consider the one parameter family of surfaces $M_d, d > 0$, that have origin in [7, Section 4] and whose geometry is described in [10, Proposition 2.1]. This family of surfaces was already used, for example, in [8, Example 2.1].
First we describe the asymptotic boundary of M_d, for $d > 1$.
Consider a horizontal geodesic γ in $\mathbb{H}^2$, with asymptotic boundary $\{p, q\}$ and let α be the closure of a connected component of $(\partial_\infty \mathbb{H}^2 \times \{0\}) \setminus (\{p, q\} \times \{0\})$. Let

$$H(d) = \int_{\cosh^{-1}(d)}^{+\infty} \frac{d}{\sqrt{\cosh^2 u - d^2}}\, du, \quad d > 1$$

be the positive number defined in (1) of [10]. Notice that $\lim_{d \longrightarrow 1} H(d) = +\infty$ and $\lim_{d \longrightarrow +\infty} H(d) = \pi/2$.
Let α_d in $\partial_\infty \mathbb{H}^2 \times \{H(d)\}$ and α_{-d} in $\partial_\infty \mathbb{H}^2 \times \{-H(d)\}$ be the two curves that project vertically onto α. Let L_d, R_d be two vertical segments in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$ of height $2H(d)$ such that the curve $L_d \cup \alpha_d \cup R_d \cup \alpha_{-d}$ is a closed simple curve. Then $\partial_\infty M_d = L_d \cup \alpha_d \cup R_d \cup \alpha_{-d}$.
Now we describe the position of M_d in the ambient space, for $d > 1$.
Denote by Q_γ the halfspace determined by $\gamma \times \mathbb{R}$, whose asymptotic boundary contains the curve α. Let γ_d be the curve in $Q_\gamma \cap (\mathbb{H}^2 \times \{0\})$ at constant distance $\cosh^{-1}(d)$ from

γ. M_d contains the curve γ_d. Denote by Z_d the closure of the non mean convex side of the cylinder over the curve γ_d. Then, M_d is contained in Z_d which is contained in Q_γ. Notice that any vertical translation of the surface M_d is contained in Z_d. Moreover, any vertical translation of M_d is arbitrarily close to Q_γ if d is sufficiently close to 1.
We observe that in the description above, γ can be any geodesic of $\mathbb{H}^2$.

Proof of Theorem 3.1. The proof is an application of the maximum principle between the surface M and the one parameter family of surfaces M_d.

We choose the geodesic γ, in order to construct the M_d's, as follows. Let $\gamma \subset \mathbb{H}^2$ be any geodesic such that

- P1: The halfspace Q_γ is strictly contained in $(\mathbb{H}^2 \times \mathbb{R}) \setminus P_+$.
- P2: $\partial_\infty \gamma \cap \partial_\infty P = \emptyset$.

Now, notice that

(1) The intersection of $\partial_\infty M$ with $\partial_\infty(\mathbb{H}^2 \times \mathbb{R}) \setminus \partial_\infty P_+$ contains no points at finite height.
(2) The asymptotic boundary of any vertical translation M_d is contained in the asymptotic boundary of $Q_\gamma \subset \mathbb{H}^2 \times \mathbb{R} \setminus P_+$.

We claim that M_d and M are disjoint for any $d > 1$. Indeed, letting $p \longrightarrow q$ (recall that p, q are the endpoints of the geodesic γ), one has that M_d collapses to a vertical segment in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$. Suppose that, when $p \longrightarrow q$, the surfaces M_d always have a nonempty intersection with M. Then, there would exists a point of the asymptotic boundary of M at finite height in $\partial_\infty(\mathbb{H}^2 \times \mathbb{R}) \setminus \partial_\infty P_+$, giving a contradiction with (1). Then, if $M \cap M_d \neq \emptyset$, we would obtain a last intersection point between M and some modified M_d letting $p \longrightarrow q$, contradicting the maximum principle.

Therefore, by the maximum principle, any vertical translation of M_d and M are disjoint.
Let $d \longrightarrow 1$. By the maximum principle, there is no first point of contact between M_d and M. As we can apply the maximum principle between any vertical translation of M_d and M, one has that M is contained in the closed halfspace $\mathbb{H}^2 \times \mathbb{R} \setminus Q_\gamma$ for any geodesic γ satisfying the properties P1 and P2. Therefore, M is included in the closure of P_+.

Now we have one of the following possibilities:

- Some points of the interior of M touches $\partial P_+ = P$, then, by the maximum principle, $M \subset P$.
- $M \setminus \partial M$ is contained in the halfspace P_+.

The result is thus proved. □

Let us give a definition, before stating some consequences of Theorem 3.1.

Definition 3.2. We say that $L \subset \partial_\infty(\mathbb{H}^2 \times \mathbb{R})$ is a *line* if $L = \{p\} \times \mathbb{R}$ for some $p \in \partial_\infty \mathbb{H}^2$.
Given vertical lines $L_1, \ldots, L_k$ in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$, we define the set $P(L_1, \ldots, L_k)$ as follows. Let P_i the vertical plane such that $\partial_\infty P_i \cap (\partial_\infty \mathbb{H}^2 \times \mathbb{R}) = L_i \cup L_{i+1}$ (with the convention

that $L_{k+1} = L_1$). Denote by $\tilde{P}_i$ the halfspace determined by the vertical plane P_i such that $\bigcup_j L_j \subset \partial_\infty \tilde{P}_i$. Then, we set $P(L_1, \dots, L_k) := \cap_i \tilde{P}_i$.

Corollary 3.1. *Let M be a complete minimal surface, possibly with finite boundary, properly immersed in $\mathbb{H}^2 \times \mathbb{R}$ and let $\Gamma = \partial_\infty M \cap (\partial_\infty \mathbb{H}^2 \times \mathbb{R})$. Let $L_1, \dots, L_k$ be vertical lines in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$. If $\Gamma \subset L_1 \cup \dots \cup L_k$ and $\partial M \subset \overline{P(L_1, \dots, L_k)}$, then $M \setminus \partial M$ is contained in $P(L_1, \dots, L_k)$, unless M is contained in one of the P_i.*

Proof. By Theorem 3.1, M is contained in every halfspace $\tilde{P}_i$ determined by the vertical plane P_i such that $\bigcup_j L_j \subset \partial_\infty \tilde{P}_i$, unless it is contained in one of the P_i. Hence it is contained in $P(L_1, \dots, L_k)$, by definition, unless it is contained in one of the P_i. □

Corollary 3.2. *Let M be a complete minimal surface properly immersed in $\mathbb{H}^2 \times \mathbb{R}$. Let P be a vertical plane. If $\partial_\infty M \cap (\partial_\infty \mathbb{H}^2 \times \mathbb{R}) \subset \partial_\infty P$, then $M = P$.*

Proof. By Theorem 3.1, M is contained in the closure of both halfspaces determined by P, hence it is contained in P. Then $M = P$ because it is complete. □

Corollary 3.3. *Let M be a complete minimal surface properly immersed in $\mathbb{H}^2 \times \mathbb{R}$. Suppose that the asymptotic boundary of M is contained in the asymptotic boundary of a totally geodesic plane S of $\mathbb{H}^2 \times \mathbb{R}$. Then $M = S$.*

Proof. The proof is a simple consequence of the maximum principle and of the previous results. We do it for completeness. First assume that the asymptotic boundary of M is contained in the asymptotic boundary of a slice, say $\{t = 0\}$. Then, for n sufficiently large, the slice $\{t = n\}$ is disjoint from M. Now, we translate the slice $\{t = n\}$ down. The first contact point, cannot be interior because of the maximum principle, hence M must stay below the slice $\{t = 0\}$. One can do the same reasoning with slices coming from the bottom, and M must stay above the slice $\{t = 0\}$. Hence M coincides with the slice $\{t = 0\}$.
If the the asymptotic boundary of M is contained is the asymptotic boundary of a vertical plane, the result follows by Corollary 3.2. □

Corollary 3.4. *Let M be a complete minimal surface properly immersed in $\mathbb{H}^2 \times \mathbb{R}$. Assume that the projection of the asymptotic boundary of M into $\partial_\infty \mathbb{H}^2$ omits a closed interval α joining two points p and q. Let γ be the horizontal geodesic in $\mathbb{H}^2$ whose the asymptotic boundary is $\{p, q\}$ and let Q_γ be the halfspace determined by $\gamma \times \mathbb{R}$ whose asymptotic boundary contains α. Then M is contained in $\mathbb{H}^2 \times \mathbb{R} \setminus \overline{Q_\gamma}$.*

Proof. By hypothesis $\partial_\infty M \cap (\partial_\infty \mathbb{H}^2 \times \mathbb{R})$ is contained in the asymptotic boundary of $(\mathbb{H}^2 \times \mathbb{R}) \setminus Q_\gamma$. The result follows by Theorem 3.1 with $P_+ = (\mathbb{H}^2 \times \mathbb{R}) \setminus \overline{Q_\gamma}$. □

Remark 3.1. There exist examples of minimal surfaces with asymptotic boundary equal to two vertical halflines, lines and a curve at finite height, see [7, Equation (32)] and [10, Proposition 2.1 (2)].

4. Some generalizations to $\mathbb{H}^n \times \mathbb{R}$.

Let us recall the construction and the properties of the n-catenoids in $\mathbb{H}^n \times \mathbb{R}$, $n \geqslant 3$, established, by P. Bérard and the second author in [2, Proposition 3.2]. Given any $a > 0$ we denote by $\big(I_a, f(a,\cdot)\big)$, where $I_a \subset \mathbb{R}$ is an interval, the maximal solution of the following Cauchy problem:

$$\begin{cases} f_{tt} = (n-1)(1+f_t^2)\coth(f), \\ f(0) = a > 0, \\ f_t(0) = 0. \end{cases}$$

Theorem 4.1 ([2]). *For $a > 0$, the maximal solution $\big(I_a, f(a,\cdot)\big)$ gives rise to the generating curve C_a, parametrized by $t \mapsto \big(\tanh(f(a,t)), t\big)$, of a complete minimal rotational hypersurface $\mathcal{C}_a$* (n-catenoid) *in $\mathbb{H}^n \times \mathbb{R}$, with the following properties.*

(1) *The interval I_a is of the form $I_a =]-T(a), T(a)[$ where*
$$T(a) = \sinh^{n-1}(a) \int_a^\infty \big(\sinh^{2n-2}(u) - \sinh^{2n-2}(a)\big)^{-1/2}\, du.$$
(2) *$f(a,\cdot)$ is an even function of the second variable.*
(3) *For all $t \in I_a$, $f(a,t) \geq a$.*
(4) *The derivative $f_t(a,\cdot)$ is positive on $]0, T(a)[$, negative on $]-T(a), 0[$.*
(5) *The function $f(a,\cdot)$ is a bijection from $[0, T(a)[$ onto $[a, \infty[$, with inverse function $\lambda(a,\cdot)$ given by*
$$\lambda(a,\rho) = \sinh^{n-1}(a) \int_a^\rho \big(\sinh^{2n-2}(u) - \sinh^{2n-2}(a)\big)^{-1/2}\, du.$$
(6) *The catenoid $\mathcal{C}_a$ has finite vertical height $h_R(a) := 2T(a)$,*
(7) *The function $a \mapsto h_R(a)$ increases from 0 to $\frac{\pi}{(n-1)}$ when a increases from 0 to infinity. Furthermore, given $a \neq b$, the generating catenaries C_a and C_b intersect at exactly two symmetric points.*

For later use, we need the following result. Altough we believe that the result is classical, we give a proof for the sake of completeness. The reader is referred to [4, chapter VII] or [13, chapter 9, addendum 3] for the proof of the analogous statement in Euclidean space.

Proposition 4.1. *Let $S \subset \mathbb{H}^n$ be a finite union of connected, closed and embedded $(n-1)$-submanifolds C_j, $j = 1, \dots, k$, such that the bounded domains whose boundary are the C_j are pairwise disjoint. Assume that for any geodesic $\gamma \subset \mathbb{H}^n$, there exists a $(n-1)$-geodesic plane $\pi_\gamma \subset \mathbb{H}^n$ of symmetry of S which is orthogonal to γ. Then S is a $(n-1)$-geodesic sphere of $\mathbb{H}^n$.*

Proof. We will proceed the proof by induction on $n \geqslant 2$.
First assume that $n = 2$. By hypothesis, there exist two geodesics $c_1, c_2 \subset \mathbb{H}^2$ of symmetry of the closed curve S intersecting at some point $p \in \mathbb{H}^2$ and making an angle

$\alpha \neq 0$ such that π/α is not rational. For any $q \in S$, denote by C_q the circle centered at p passing through q. Then C_q is contained in S. Let $\widetilde{q} \neq q$ be points of S. If $C_q \neq C_{\widetilde{q}}$ then the geodesic disks bounded by C_q and $C_{\widetilde{q}}$ are not disjoint, since they have the same center, which contradicts the hypothesis. Consequently, we get $C_q = C_{\widetilde{q}}$ and we conclude that S is a circle.

Let $n \in \mathbb{N}$, $n \geqslant 3$. Assume that the statement holds for $k = 2, \ldots, n-1$.
Let $\pi_0 \subset \mathbb{H}^n$ be a $(n-1)$-geodesic plane of symmetry of S.

Claim 1. $S \cap \pi_0$ is a $(n-2)$-geodesic sphere of π_0.

Indeed, let $\gamma \subset \pi_0$ be a geodesic. By hypothesis there exists a $(n-1)$-geodesic plane $\pi_\gamma \subset \mathbb{H}^n$ orthogonal to γ which is a plane of symmetry of S. Since π_γ is orthogonal to π_0, then $S \cap \pi_0$ is symmetric about $\pi_\gamma \cap \pi_0$ (which is a $(n-2)$-geodesic plane of π_0), see [11, Lemme 3.3.15]. As π_0 is a $(n-1)$ hyperbolic space, $S \cap \pi_0$ satisfies the assumptions of the statement in $\mathbb{H}^{n-1}$.
By the induction hypothesis we deduce that $S \cap \pi_0$ is a $(n-2)$-geodesic sphere of π_0. This proves Claim 1.
Let $p_0 \in \pi_0$ and $\rho_0 > 0$ be respectively the center and the radius of the $(n-2)$-geodesic sphere $S \cap \pi_0$.

Claim 2. Let $\pi_1 \subset \mathbb{H}^n$ be a $(n-1)$-geodesic plane of symmetry of S orthogonal to π_0. Then $S \cap \pi_1$ is a $(n-2)$-geodesic sphere of π_1 with center p_0 and radius ρ_0.

Claim 1 yields that $S \cap \pi_1$ is a $(n-2)$-geodesic sphere of π_1. Since π_0 and π_1 are orthogonal, then the geodesic sphere $S \cap \pi_0$ is symmetric about π_1. Therefore $p_0 \in \pi_1$. If $n > 3$, then $(S \cap \pi_0) \cap \pi_1$ is $(n-3)$-geodesic sphere with center p_0 and radius ρ_0 of $\pi_0 \cap \pi_1$ (which is a $(n-2)$ hyperbolic space). If $n = 3$, then $(S \cap \pi_0) \cap \pi_1$ is constituted of two points whose the distance is $2\rho_0$. In both cases we infer that $\operatorname{diam}_{\mathbb{H}^n}(S \cap \pi_1) \geqslant 2\rho_0$ and then the radius of the geodesic sphere $S \cap \pi_1$ is $\rho_1 \geqslant \rho_0$. Analogously we can show that $\rho_0 \geqslant \rho_1$. We deduce that $\rho_1 = \rho_0$, that is $S \cap \pi_0$ and $S \cap \pi_1$ have both center at p_0 and radius ρ_0. This proves Claim 2.

Claim 3. Let $\pi_2 \subset \mathbb{H}^n$ be any $(n-1)$-geodesic plane of symmetry of S. Then $S \cap \pi_2$ is a $(n-2)$-geodesic sphere of π_2 with center p_0 and radius ρ_0.

Since S is symmetric with respect to π_0 and π_2, π_0 and π_2 are distinct and S is compact, then the $(n-1)$-geodesic planes π_0 and π_2 cannot be disjoint.
Then, we find a third $(n-1)$-geodesic plane π_3 of symmetry of S, orthogonal to both π_0 and π_2. Claim 2 implies that $S \cap \pi_2$ is a $(n-2)$-geodesic sphere of π_2 with center p_0 and radius ρ_0. This proves Claim 3.

Now we finish the proof of the Proposition as follows. Let $p \in S$ and let $\pi \subset \mathbb{H}^n$ be any $(n-1)$-geodesic plane passing through p and p_0. Let $\gamma \subset \mathbb{H}^n$ be the geodesic through p_0 orthogonal to π. By Claim 2, there exists a $(n-1)$-geodesic plane π_γ of symmetry of S and orthogonal to γ. Claim 3 ensures that $p_0 \in \pi_\gamma$, then $\pi_\gamma = \pi$. Claim 3 yields also that $S \cap \pi$ is $(n-2)$-geodesic sphere of π with center p_0 and radius ρ_0,

thus $d_{\mathbb{H}^n}(p,p_0)=\rho_0$. This shows that S is the $(n-1)$-geodesic sphere of $\mathbb{H}^n$ of radius ρ_0 and center p_0. □

Now we establish a characterization of the n-catenoid, that is a generalization to higher dimension of Theorem 2.1.

Theorem 4.2. *Let $M \subset \mathbb{H}^n \times \mathbb{R}$ be an immersed, connected, complete minimal hypersurface with two ends. Assume that each end is a vertical graph whose asymptotic boundary is a copy of $\partial_\infty \mathbb{H}^n$. Then M is a n-catenoid.*

Proof. Up to a vertical translation, we can assume that the asymptotic boundary of M is symmetric with respect to $\Pi := \mathbb{H}^n \times \{0\}$. We set $\Gamma^+ := \partial_\infty M \cap \{t > 0\}$ and recall that Γ^+ is a copy of $\partial_\infty \mathbb{H}^n$. As usual we set $M^+ := M \cap \{t > 0\}$.
Next Claim can be shown in the same fashion as in $\mathbb{H}^2 \times \mathbb{R}$ (see Lemma 2.1 and the proof of Claim 2 of Theorem 2.1). For this reason we just state it.
Claim. M is symmetric about Π, and each connected component of $M \setminus \Pi$ is a vertical graph. Moreover, for any geodesic $\gamma \subset \Pi$ there exists a vertical hyperplane $P_\gamma \subset \mathbb{H}^n \times \mathbb{R}$ orthogonal to γ which is a n-plane of symmetry of M. Therefore, $\pi_\gamma := P_\gamma \cap \Pi$ is a $(n-1)$-plane of symmetry of $\Sigma := M \cap \Pi$.
Using the result of the Claim we get that Σ satisfies the assumptions of Proposition 4.1. Then Σ is a $(n-1)$-geodesic sphere of Π, since $\Pi = \mathbb{H}^n \times \{0\}$.
Let $\mathcal{C} \subset \mathbb{H}^n \times \mathbb{R}$ be the catenoid through Σ and orthogonal to Π. We set $\mathcal{C}^+ := \mathcal{C} \cap \{t > 0\}$.
Both $\mathcal{C}^+$ and M^+ are vertical along their common finite boundary Σ, hence they are tangent along Σ.
Let $t_{\mathcal{C}}$ (resp. t_M) the height of the asymptotic boundary of $\mathcal{C}^+$ (resp. M^+).
Suppose for example that $t_{\mathcal{C}} \leqslant t_M$. Then, lifting upward and downward M^+, we obtain that M^+ is above $\mathcal{C}^+$. Therefore we deduce that $M^+ = \mathcal{C}^+$ by applying the boundary maximum principle. The case $t_M \leqslant t_{\mathcal{C}}$ is analogous.
We conclude that $M = \mathcal{C}$ and the proof is completed. □

In order to establish the generalization in higher dimension of Theorem 3.1, we need to state some existence results, established for $n \geqslant 3$, in [2, Theorem 3.8], inspired by [10, Proposition 2.1]. In fact, we only need the $d > 1$ case, but we state the whole result for the sake of completeness.

Theorem 4.3 ([2])**.** *There exists a one parameter family $\{\mathcal{M}_d,\ d > 0\}$ of complete embedded minimal hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$ invariant under hyperbolic translations.*

1. *If $d > 1$, then $\mathcal{M}_d$ consists of the union of two symmetric vertical graphs over the exterior of an equidistant hypersurface in the slice $\mathbb{H}^n \times \{0\}$.*

   The asymptotic boundary of $\mathcal{M}_d$ is topologically an $(n-1)$-sphere which is homologically trivial in $\partial_\infty \mathbb{H}^n \times \mathbb{R}$. More precisely, we set for $d > 1$:

$$S(d) = \cosh(a) \int_1^\infty (t^{2n-2}-1)^{-1/2} (\cosh^2(a) t^2 - 1)^{-1/2}\, dt, \quad \textit{where } d =: \cosh^{n-1}(a).$$

Then, the asymptotic boundary of $\mathcal{M}_d$ consists of the union of two copies of an hemisphere $S_+^{n-1} \times \{0\}$ of $\partial_\infty \mathbb{H}^n \times \{0\}$ in parallel slices $t = \pm S(d)$, glued with the finite cylinder $\partial S_+^{n-1} \times [-S(d), S(d)]$

The vertical height of $\mathcal{M}_d$ is $2S(d)$. The height of the family $\mathcal{M}_d$ is a decreasing function of d and varies from infinity (when $d \to 1$) to $\pi/(n-1)$ (when $d \to \infty$).

(2) *If $d = 1$, then $\mathcal{M}_1$ consists of a complete (non-entire) vertical graph over a halfspace in $\mathbb{H}^n \times \{0\}$, bounded by a totally geodesic hyperplane P. It takes infinite boundary value data on P and constant asymptotic boundary value data. The asymptotic boundary of $\mathcal{M}_1$ is the union of a spherical cap S of $\partial_\infty \mathbb{H}^n \times \{c\}$ with a half vertical cylinder over ∂S.*

(3) *If $d < 1$, then $\mathcal{M}_d$ is an entire vertical graph with finite vertical height. Its asymptotic boundary consists of a homologically non-trivial $(n-1)$-sphere in $\partial_\infty \mathbb{H}^n \times \mathbb{R}$.*

The hypersurfaces $\mathcal{M}_d$ are the analogous in higher dimension of the surfaces M_d in $\mathbb{H}^2 \times \mathbb{R}$. Also, as in $\mathbb{H}^2 \times \mathbb{R}$, by (*vertical*) *hyperplane* we mean a complete totally geodesic hypersurface $\Pi \times \mathbb{R}$, where Π is any totally geodesic hyperplane of $\mathbb{H}^n \times \{0\}$. Moreover, we call a *vertical halfspace* any component of $(\mathbb{H}^n \times \mathbb{R}) \setminus P$ where P is a vertical hyperplane. Thus, working with the hypersurfaces $\mathcal{M}_d$ exactly in the same way as in Theorem 3.1, we obtain the following result.

Theorem 4.4. *Let M be a complete minimal hypersurface properly immersed in $\mathbb{H}^n \times \mathbb{R}$, possibly with finite boundary. Let P be a vertical geodesic hyperplane and P_+ one of the two halfspaces determined by P. If $\partial M \subset \overline{P_+}$ and $\partial_\infty M \cap (\partial_\infty \mathbb{H}^n \times \mathbb{R}) \subset \partial_\infty P_+$, then $M \setminus \partial M \subset P_+$, unless $M \subset P$.*

Obviously, the analogous in higher dimension of Corollaries 3.1, 3.2, 3.3 hold as well. Part (1) of next Theorem is a generalization in higher dimension of Corollary 3.4, while part (2) was proved, for $n = 2$ by the second and the third authors [10, Corollary 2.2]

Theorem 4.5. *Let $S_\infty \subset \partial_\infty \mathbb{H}^n \times \mathbb{R}$ be a closed set whose the vertical projection on $\partial_\infty \mathbb{H}^n \times \{0\}$ omits an open subset U.*

(1) *Let M be a complete minimal hypersurface properly immersed in $\mathbb{H}^n \times \mathbb{R}$ such that $\partial_\infty M = S_\infty$. Let $Q \subset \mathbb{H}^n \times \mathbb{R}$ be a vertical halfspace whose asymptotic boundary is contained in $U \times \mathbb{R}$. Then M is contained in $\mathbb{H}^n \times \mathbb{R} \setminus \overline{Q}$.*

(2) *Assume that S_∞ is contained in an open slab whose height is equal to $\frac{\pi}{n-1}$. Then, there is no complete connected properly immersed minimal hypersurface M in $\mathbb{H}^n \times \mathbb{R}$ with asymptotic boundary S_∞.*

Proof. The first statement is a consequence of Theorem 4.4 and the proof is analogous to that of Corollary 3.4.

Let us prove the second statement. Assume, by contradiction, that there is such a minimal hypersurface M with asymptotic boundary S_∞. Then, up to a vertical

translation, we can assume that M is contained in the slab $\mathcal{S} := \{\varepsilon < t < \frac{\pi}{n-1} - \varepsilon\}$ for some $\varepsilon > 0$, and thus $S_\infty \subset \partial_\infty \mathcal{S}$. By assumption, there exists a $(n-1)$-geodesic plane $\pi \subset \mathbb{H}^n \times \{0\}$ such that a component π^+ of $\mathbb{H}^n \times \{0\} \setminus \pi$ satisfies:

(1) $\partial_\infty \pi^+ \subset U$.
(2) $M \cap (\pi^+ \times \mathbb{R}) = \emptyset$.

Let $C \subset \mathbb{H}^n \times (0, \frac{\pi}{n-1})$ be any n-catenoid such that a component of its asymptotic boundary stays stricly above $\partial_\infty \mathcal{S}$ and the other component stays strictly below $\partial_\infty \mathcal{S}$. We take a connected and compact piece K of C such that its boundary lies in the boundary of the slab $\mathcal{S}$.
Let $q \in M$ be a point and let $q_0 \in \mathbb{H}^n \times \{0\}$ be the vertical projection of q. Let $p_\infty \in \partial_\infty \pi^+$ be an asymptotic point. Denote by $\widetilde{\gamma} \subset \partial_\infty \mathbb{H}^n \times \{0\}$ the complete geodesic passing through q_0 such that $p_\infty \in \partial_\infty \widetilde{\gamma}$. We can translate K along $\widetilde{\gamma}$ such that the translated K is contained in the halfspace $\pi^+ \times \mathbb{R}$.
Now we come back translating K towards M along $\widetilde{\gamma}$. Observe that the boundary of the translated copies of K does not touch M. Therefore, doing the translations of K along $\widetilde{\gamma}$ we find a first interior point of contact between M and a translated copy of K. Hence, $M = C$ by the maximum principle, which leads to a contradiction. This completes the proof. □

Now we state a generalization of the Asymptotic Theorem proved in [10, Theorem 2.1]. Our result establishes some obstruction for the asymptotic boundary of a complete properly immersed minimal hypersurface in $\mathbb{H}^n \times \mathbb{R}$.

Theorem 4.6 (Asymptotic Theorem). *Let $\Gamma \subset \partial_\infty \mathbb{H}^n \times \mathbb{R}$ be a connected $(n-1)$-submanifold with boundary. Let $\mathrm{Pr} : \partial_\infty \mathbb{H}^n \times \mathbb{R} \to \partial_\infty \mathbb{H}^n$ be the projection on the first factor. Assume that:*

(1) There is some point $q_\infty \in \partial\,\mathrm{Pr}(\Gamma)$ such that $q_\infty \notin \mathrm{Pr}(\partial\Gamma)$.
(2) $\Gamma \subset \partial_\infty \mathbb{H}^n \times (t_0, t_0 + \frac{\pi}{n-1})$ for some real number t_0.

Then, there is no properly and completely immersed minimal hypersurface (maybe with finite boundary) $M \subset \mathbb{H}^n \times \mathbb{R}$ such that $\partial_\infty M = \Gamma$ and $M \cup \Gamma$ is a continuous n-manifold with boundary.

Proof. Assume, by contradiction, that there is such a minimal hypersurface M. Since $q_\infty \in \partial \mathrm{Pr}(\Gamma)$ and $q_\infty \notin \mathrm{Pr}(\partial\Gamma)$, there exists a $(n-1)$-geodesic plane $\omega \subset \mathbb{H}^n \times \{0\}$ such that a component ω^+ of $\mathbb{H}^n \times \{0\} \setminus \omega$ satisfies:

(1) $q_\infty \in \partial_\infty \omega^+$, $q_\infty \notin \partial_\infty \omega$ and $\partial_\infty \omega^+ \cap \mathrm{Pr}(\partial\Gamma) = \emptyset$.
(2) If M_0 denotes the component of $M \cap (\omega^+ \times \mathbb{R})$ containing q_∞ in its asymptotic boundary, then
   (a) $M_0 \subset \mathbb{H}^n \times (t_0, t_0 + \frac{\pi}{n-1})$ for some real number t_0.
   (b) $\partial M_0 \subset \omega \times (t_0 + 2\varepsilon, t_0 - 2\varepsilon + \frac{\pi}{n-1})$ for some $\varepsilon > 0$.

Again, since $q_\infty \in \partial \mathrm{Pr}(\Gamma)$ and $q_\infty \notin \mathrm{Pr}(\partial\Gamma)$, there exists a $(n-1)$-geodesic plane $\pi \subset \mathbb{H}^n \times \{0\}$ such that a component π^+ of $\mathbb{H}^n \times \{0\} \setminus \pi$ satisfies:

(1) $\pi^+ \subset \omega^+$.
(2) $\partial_\infty \pi^+ \cap$ Pr$(\Gamma) = \emptyset$.
(3) $M_0 \cap (\pi^+ \times \mathbb{R}) = \emptyset$.

Therefore we can find a compact part K of a n-catenoid satisfying:

(1) K is connected.
(2) $K \subset \pi^+ \times (t_0 + \varepsilon, t_0 - \varepsilon + \frac{\pi}{n-1})$.
(3) $\partial K \subset \mathbb{H}^n \times \{t_0 + \varepsilon, t_0 - \varepsilon + \frac{\pi}{n-1}\}$.

We deduce consequently that $M_0 \cap K = \emptyset$. Then, considering the horizontal translated copies of K and arguing as in the proof of Theorem 4.5, we get a contradiction with the maximum principle, which concludes the proof. □

The following result is an immediate consequence of Theorem 4.6.

Corollary 4.1. *Let $S_\infty \subset \partial_\infty \mathbb{H}^n \times \mathbb{R}$ be a $(n-1)$-closed continuous submanifold. Considering the halfspace model for $\mathbb{H}^n$, we can assume that $S_\infty \subset \mathbb{R}^{n-1} \times \mathbb{R}$.*
If S_∞ is strictly convex in Euclidean sense, then there is no complete connected properly immersed minimal hypersurface M in $\mathbb{H}^n \times \mathbb{R}$, possibly with finite boundary, with asymptotic boundary S_∞ and such that $M \cup S_\infty$ is a continuous n-manifold with boundary.

Remark 4.1. It follows from Corollary 4.1 that there is no horizontal minimal graph in $\mathbb{H}^n \times \mathbb{R}$, [9, Equation (3)], given by a positive function $g \in C^2(\Omega) \cap C^0(\overline{\Omega})$, where $\Omega \subset \mathbb{R}^{n-1} \times \mathbb{R} \subset \partial_\infty \mathbb{H}^n \times \mathbb{R}$ is a bounded strictly convex domain in Euclidean sense, assuming zero value on $\partial\Omega$.

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